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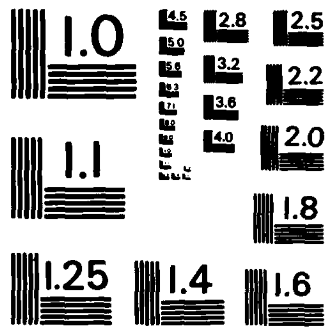

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CONTINUITY OF CERTAIN RANDOM INTEGRAL
MAPPINGS AND THE UNIFORM INTEGRABILITY
OF INFINITELY DIVISIBLE MEASURES

by

Zbigniew J. Jurek and Jan Rosinski

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CONTINUITY OF CERTAIN RANDOM INTEGRAL MAPPINGS AND THE UNIFORM INTEGRABILITY OF INFINITELY DIVISIBLE MEASURES

by

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Abstract: In this paper it is shown that the class of probability measures L(Q), which generalizes the classical Lévy class L, is homeomorphic with the class ID_{log} of all infinitely divisible probability measures having finite logarithmic moment. As an application of this result a set of generators of the entire class L(Q) is described. As a necessary tool, the relationship between the uniform integrability of infinitely divisible measures and of their corresponding Lévy measures is studied and this may be of independent interest.

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Keywords: infinitely divisible probability measures on Banach spaces, operator--selfdecomposable measures, random integrals

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1. Preliminaries and notations.

Let E be a real separable Banach space with norm $||\cdot||$ and Borel σ-field $\mathcal{B}$. Let $\mathcal{P}(E)$ be the set of all probability measures on $\mathcal{B}$ which, with the convolution "*" and the weak convergence topology, becomes a topological semigroup. By $\mu_n \Rightarrow \mu$ we denote the weak convergence of μ_n to μ as $n\to\infty$, for $\mu_n, \mu \in \mathcal{P}(E)$. A $\mu \in \mathcal{P}(E)$ is said to be infinitely divisible if for every integer $n \geq 2$ there is $\mu_n \in \mathcal{P}(E)$ such that $\mu_n^{*n} = \mu$. The class $ID(E)$ of all infinitely divisible measures on E is a closed subsemigroup of $\mathcal{P}(E)$. Furthermore, each $\mu \in ID(E)$ is uniquely determined by the triplet: a vector $a \in E$, a Gaussian covariance operator R and a Lévy measure M, cf. Araujo and Giné (1980). In this case we write $\mu = [a,R,M]$; note that Araujo and Giné (1980) use the notation: $\mu = \delta(a)*\gamma*c_1\mathrm{Pois}(M)$, where γ is a symmetric Gaussian measure with the covariance operator R.

Let Q be a fixed bounded linear operator on E such that $\lim\limits_{t\to\infty} \exp(-tQ) = 0$ in the operator topology. It is easy to see that Q is an isomorphism on E. Indeed, since the function $t \to ||e^{-tQ}||$ is submultiplicative and vanishes at $+\infty$ there are positive constants a and b such that $||e^{-tQ}|| \leq ae^{-bt}$ for all $t > 0$. Hence Bochner integral $\int_0^\infty e^{-tQ}dt$ exists and is equal to the inverse operator to Q. Now we define

$$L(Q) := \{\mu \in \mathcal{P}(E)\colon \forall(t>0)\ \exists\ (\mu_t \in \mathcal{P}(E))\ \mu=e^{-tQ}\mu*\mu_t\}$$

(recall that for a measure ν on E and a Borel measurable function f on E the measure $f\nu$ is defined by $(f\nu)(B) = \nu(f^{-1}(B))$ for all $B \in \mathcal{B}$). Observe that $L(Q)$ coincides with the Lévy class L of all asymptotic distributions of partial sums of independent random variables, if $Q = Id$ is the identity operator on E. In general, $L(Q)$ is a class of limit distributions of partial sums of independent random variables normed by linear bounded operators on E, cf. Urbanik (1978). Moreover $L(Q)$ is a closed subsemigroup of $ID(E)$.

Let $D_E[0,\infty)$ denote the set of all functions from $[0,\infty)$ into E which are right continuous and have left limits. The topology in $D_E[0,\infty)$ is defined as in Lindwall (1972) (cf. also Gikhman and Skorohod (1974)) and $D_E[0,\infty)$ equipped with such a topology is a separable metric space. For a $D_E[0,\infty)$-valued random variable $X = \{X(t):t\geq 0\}$ and the operator valued function $t \to e^{tA}$, where A is a bounded linear operator on E, we define random integral as follows:

$$\int_{(a,b]} e^{tA}dX(t) := e^{bA}X(b) - e^{aA}X(a) - \int_{(a,b]} d(e^{tA})X(t)$$

$$:= e^{bA}X(b) - e^{aA}X(a) - \int_{(a,b]} (e^{tA}A)X(t)dt,$$

$0 \leq a < b < \infty$, where the last stochastic integral is defined path-wise; cf. Jurek (1982) and Jurek and Vervaat (1983).

Let Q be as above. Jurek (1982) has shown that $\nu \in L(Q)$ if and only if there exists a $D_E[0,\infty)$-valued random variable X with independent and stationary increments, $X(0) = 0$ and $E \log (1 + ||X(1)||) < \infty$ such that

(1) $$\nu = \mathcal{L}\left(\int_{(0,\infty)} e^{-tQ}dX(t)\right),$$

where $\mathcal{L}(\xi)$ denotes the distribution of a random variable ξ. Here

$\int_{(0,\infty)} e^{-tQ}dX(t)$ is defined as the limit a.s. (or in probability) of

$\int_{(0,b]} e^{-tQ}dX(t)$ as $b \to \infty$, and this limit exists if and only if $E \log (1+||X(1)||)<\infty$;

cf. Jurek (1982). Let

$$ID_{\log} := ID_{\log}(E) := \{\mu \in ID(E): \int_E \log(1+||x||)\mu(dx) < \infty\}.$$

Since the distribution of X is determined by $\mathcal{L}(X(1)) = \mu \in ID_{\log}$, the equation (1) can be rewritten as

(2) $$\nu = J_Q(\mu) := \mathcal{L}\left(\int_{(0,\infty)} e^{-tQ}dX(t)\right).$$

The mapping J_Q is an algebraic isomorphism between the semigroups $ID_{\log}$ and $L(Q)$, and its fixed-points are the operator-stable measures (stable measures if Q is the identity operator), cf. Jurek (1982). The aim of this paper is to describe the topological properties of J_Q and to find elements of $L(Q)$ which generate, by taking convolutions and weak limits, the entire class $L(Q)$.

2. Main results.

Let $\mu_n \in \mathcal{P}(E)$ be such that $\int_E \log(1 + ||x||)\mu_n(dx) < \infty$, $n \in \mathbb{N} \cup \{0\}$. We say that μ_n *log-converges to* μ_0, and write $\mu_n \Rightarrow_{\log} \mu_0$, if $\mu_n \Rightarrow \mu_0$ and $\lim_{n\to\infty} \int_E \log(1 + ||x||)\mu_n(dx) = \int_E \log(1 + ||x||)\mu_0(dx)$. Using Billingsley (1968) Theorems 5.1 and 5.4 it is easy to deduce that $\mu_n \Rightarrow_{\log} \mu_0$ if and only if $\mu_n \Rightarrow_{\log} \mu_0$ and $\lim_{t\to\infty} \sup_{n\in\mathbb{N}} \int_{\{||x|| > t\}} \log(1 + ||x||)\mu_n(dx) = 0$.

Theorem 1. *Let* X_n, $n \in \mathbb{N} \cup \{0\}$ *be* $D_E[0,\infty)$*-valued random variables with stationary independent increments,* $X_n(0) = 0$ *and* $L(X_n(1)) \in ID_{\log}$. *Then, as* $n \to \infty$,

$$L\left(\int_{(0,\infty)} e^{-tQ} dX_n(t)\right) \Rightarrow L\left(\int_{(0,\infty)} e^{-tQ} dX_0(t)\right)$$

if and only if

$$L(X_n(1)) \Rightarrow_{\log} L(X_0(1)).$$

In other words, $J_Q(\mu_n) \Rightarrow J_Q(\mu_0)$ *for some (each)* Q *if and only if* $\mu_n \Rightarrow_{\log} \mu_0$.

Corollary 1. *Under the assumptions of Theorem* 1 $L(\int_{(0,\infty)} e^{-tQ} dX_n(t)) \Rightarrow L(\int_{(0,\infty)} e^{-tQ} dX_0(t))$ *if and only if* $L(X_n(1)) \Rightarrow L(X_0(1))$ *and the sequence* $\{\log(1 + ||X_n(1)||)\}_{n=1}^{\infty}$ *is uniformly integrable.*

Since conditions for the weak convergence of infinitely divisible measures are usually given in terms of the corresponding triplets we shall likewise characterize the uniform integrability of $\{\log(1 + ||X_n(1)||)\}$. Because in our approach only certain properties of the logarithmic function are essential we shall, in fact, prove a more general theorem.

Let Φ be the class of all continuous functions $\phi: E \to (0,\infty)$ such that for every $x,y \in E$

$$\phi(x + y) \leq c\phi(x)\phi(y),$$

where $c = c(\phi)$ is a positive constant.

Note that if $\psi: E \to [0,\infty)$ is a subadditive continuous function i.e. $\psi(x + y) \leq d[\psi(x) + \psi(y)]$ for all $x,y \in E$ and some $d \geq 1$, then $\phi(x) := d+\psi(x)$ belongs to Φ with $c = 1$. The following are examples of such functions: $\phi(x) = \exp(\lambda \; ||x||)$, $\lambda > 0$, $\psi(x) = ||x||^p$, $p > 0$ and $\psi(x) = \log(1 + ||x||)$.

Theorem 2. *Let* $\phi \in \Phi$ *and* $\mu_n = [a_n, R_n, M_n]$ *be infinitely divisible distributions on* E. *Assume that* $\{\mu_n\}$ *is relatively compact. Then*

$$\lim_{t\to\infty} \sup_n \int_{\{\phi(x) > t\}} \phi(x)\mu_n(dx) = 0$$

if and only if

$$\lim_{t\to\infty} \sup_n \int_{\{\phi(x) > t\}} \phi(x)M_n(dx) = 0.$$

Note that de Acosta and Giné (1979) (cf. also de Acosta (1980) studied the uniform integrability of similar functions in the context of the General Central Limit Theorem in Banach spaces. Our proof uses some of their arguments and results.

Corollary 2. *Under assumptions of Theorem* 1

$$L\left(\int_{(0,\infty)} e^{-tQ}dX_n(t)\right) \Rightarrow L\left(\int_{(0,\infty)} e^{-tQ}dX(t)\right)$$

if and only if

$$\mathcal{L}(X_n(1)) = [a_n, R_n, M_n] \Rightarrow [a,R,M] = \mathcal{L}(X(1))$$

and

$$\lim_{t\to\infty} \sup_n \int_{\{||x|| > t\}} \log(1 + ||x||)\, dM_n(x) = 0.$$

In the case $E = \mathbb{R}^d$, Corollary 2 has been proven by Sato and Yamazato (1984). Their proof uses, in an essential way, arguments of $\mathbb{R}^d$ and is completely different from ours.

The class ID(E) of all infinitely divisible measures on E can be described as the smallest closed subsemigroup of $P(E)$ containing all symmetric Gaussian measures and all shifted compound Poisson measures of the form $[x,0,\lambda\delta(y)]$, where, $x,y \in E$ and $\lambda > 0$, cf. Araujo and Giné (1980). Using the homeomorphism J_Q: $ID_{\log} \to L(Q)$ we shall describe a set of generators of $L(Q)$.

Let S_Q be the unit sphere in E with respect to the norm given by $||x||_Q := \int_0^\infty ||e^{-tQ}x||dt$. For every $\alpha > 0$ and $z \in S_Q$ we define a measure $M_{\alpha,z}$ on $\mathcal{B}(E\setminus\{0\})$ by

$$M_{\alpha,z}(F) = \int_0^\alpha 1_F(s^Q z)s^{-1}ds, \quad F \in \mathcal{B}(E\setminus\{0\}).$$

Since $\int \min\{1,||x||\}M_{\alpha,z}(dx) < \infty$, $M_{\alpha,z}$ is a Lévy measure on E, cf. Araujo and Giné (1980), Theorem 6.3.(i). Let K_Q consist of all generalized Poissonian measures $[x,0,\lambda M_{\alpha,z}]$, $x \in E$, $\alpha > 0$, $\lambda > 0$ and $z \in S_Q$ and of all Gaussian measures $[0,R,0]$ such that $QR + RQ^*$ is a nonnegative operator.

<u>Theorem 3</u>. *The class* L(Q) *is the smallest closed subsemigroup of* ID(E) *containing the set* K_Q.

In the case when E is a Hilbert space Jurek (1982) has obtained a slightly different set of generators of L(Q) and his proof is completely different from ours.

3. Proof of Theorem 1.

The proof is preceded by some auxilliary lemmas and propositions, which may be interesting themself.

Lemma 1. *A family* $\{\xi_\alpha\}_{\alpha\in I}$ *of real random variables is uniformly integrable if and only if*

$$\lim_{T\to\infty}\sup_{\alpha\in I}\int_T^\infty P\{|\xi_\alpha| > t\} = 0.$$

Proof. The necessity follows from the inequality

$$\int_{\{|\xi_\alpha| > T\}}|\xi_\alpha|dP = TP\{|\xi_\alpha| > T\} + \int_T^\infty P\{|\xi_\alpha| > t\}dt \geq \int_T^\infty P\{|\xi_\alpha| > t\}\, dt.$$

The sufficiency we obtain as follows

$$\begin{aligned} TP\,\{|\xi_\alpha| > 2T\} &\leq \int_{\{|\xi_\alpha| > 2T\}} [|\xi_\alpha| - T]\, dP \\ &\leq \int_{\{|\xi_\alpha| > T\}} |\xi_\alpha|\, dP - TP\,\{|\xi_\alpha| > T\} \\ &= \int_T^\infty P\{|\xi_\alpha| > t\}\, dt \end{aligned}$$

and hence

$$\int_{\{|\xi_\alpha| > 2T\}} |\xi_\alpha|\, dP \leq 2\int_T^\infty P\{|\xi_\alpha| > t\}dt + \int_{2T}^\infty P\{|\xi_\alpha| > t\}\, dt \to 0$$

uniformly in α as $T \to \infty$.

Proposition 2. (Skorohod (1957), Lemma 1.4). *Let* $Z(t)$, $t \in [a,b]$ *be a stochastic process with independent increments and trajectories in* $D_E[a,b]$.

Then for every $r > 0$

$$P\{\sup_{a\le t\le b} ||Z(t) - Z(a)|| > 2r\} \le cP\{||Z(b) - Z(a)|| > r\},$$

where $c = (1 - \sup_{a\le t\le b} P\{||Z(t) - Z(b)|| > r\})^{-1}$ *is supposed to be positive.*

Lemma 3. *Let* $\psi: [0,\infty) \to [0,\infty)$ *be an increasing function such that for some* $k > 0$ *and* $x_0 \ge 0$ $\psi(2x) \le k\psi(x)$ *for all* $x \ge x_0$. *Suppose* $Z(t)$, $t \in [a,b]$ *is a stochastic process with independent increments and paths in* $D_E[a,b]$. *Furthermore assume*

(i) $\{Z_n(t) - Z_n(b): a \le t \le b, n \in \mathbb{N}\}$ *is bounded in probability;*

(ii) $\{\psi(||Z_n(b) - Z_n(a)||): n \in \mathbb{N}\}$ *is uniformly integrable.*

Then $\{\psi(\sup_{a\le t\le b} ||Z_n(t) - Z_n(a)||): n \in \mathbb{N}\}$ *is also uniformly integrable.*

Proof. From (i) we have that $P\{||Z_n(t)-Z_n(b)|| > r\} < \frac{1}{2}$ for all $r \ge r_0$, $n \in \mathbb{N}$ and $t \in [a,b]$. By Proposition 2 we get

$$P\{\sup_{a\le t\le b} ||Z_n(t) - Z_n(a)|| > 2r\} \le 2P\{||Z_n(b) - Z_n(a)|| > r\}$$

for all $r \ge r_0$ and $n \in \mathbb{N}$. Therefore for $T \ge \psi(2(r_0 \vee x_0))$ we obtain

$$\int_T^\infty P\{\psi(\sup_{a\le t\le b} ||Z_n(t) - Z_n(a)||) > u\}du$$

$$= \int_T^\infty P\{\sup_{a\le t\le b} ||Z_n(t) - Z_n(a)|| > \psi^{-1}(u)\}\, du$$

$$\le 2\int_T^\infty P\{||Z_n(b) - Z_n(a)|| > \tfrac{1}{2}\psi^{-1}(u)\}\, du$$

$$= 2 \int_T^\infty P\{\psi(2||Z_n(b) - Z_n(a)||) > u\}\, du$$

$$\leq \int_T^\infty P\{\psi(||Z_n(b) - Z_n(a)||) > uk^{-1}\}\, du$$

$$= 2k \int_{Tk^{-1}}^\infty P\{\psi(||Z_n(b) - Z_n(a)||) > u\}\, du.$$

This together with Lemma 1 completes the proof.

Remark. Both Proposition 2 and Lemma 3 hold true if one replaces the norm $||\cdot||$ by a measurable extended-valued seminorm.

Lemma 4. *Let* X_n, $n \in \mathbb{N} \cup \{0\}$ *be* $D_E[a,b]$*-valued random variables such that* $L(X_n) \Rightarrow L(X_0)$ *in* $D_E[a,b]$ *and let* A *be a linear bounded operator on* E. *Then*

$$L\left(\int_{(a,b]} e^{tA} dX_n(t)\right) \Rightarrow L\left(\int_{(a,b]} e^{tA} dX_0(t)\right).$$

Proof. Since the mapping

$$D_E[a,b] \ni y \to \int_{(a,b]} e^{tA} dy(t) := e^{tA} y(t)\Big|_{t=a}^{t=b} - \int_{(a,b]} Ae^{tA} y(t)dt \in E$$

is continuous (cf. Billingsley (1968), p. 121), the Continuous Mapping Theorem concludes the proof (cf. Billingsley (1968), Thm. 5.1).

Proof of Theorem 1. Put

$$Y_n(t) := \int_{(0,t]} e^{-sQ} dX_n(s) \quad \text{and}$$

$$Y_n(\infty) := \int_{(0,\infty)} e^{-sQ} dX(s), \quad n \in \mathbb{N} \cup \{0\}.$$

By definition, Y_n, $n \in \mathbb{N} \cup \{0\}$ are $D_E[0,\infty)$-valued random variables with independent increments. Furthermore we have

$$Y_n(+\infty) - Y_n(t) = \int_{(t,\infty)} e^{-sQ} dX_n(s) \overset{d}{=} e^{-tQ} Y_n(\infty), \quad 0 \le t < \infty;$$

$$Y_n(j+1) - Y_n(j) = e^{-jQ} \int_{(0,1]} e^{-sQ} dX_n(s+j) \overset{d}{=} e^{-jQ} Y_n(1), \; j=0,1,\ldots,$$

and for every n

$$\xi_{nj} := \int_{(0,1]} e^{-sQ} dX_n(s+j) \overset{d}{=} Y_n(1) \quad \text{are independent,}$$

$j \in \{0\} \cup \mathbb{N}$, where "$\overset{d}{=}$" means "equal in distribution". Also we have

$$||Y_n(1)|| = ||e^{-Q} X_n(1) + \int_{(0,1]} Qe^{-tQ} X_n(t) dt|| \le C \sup_{0 \le t \le 1} ||X_n(t)||,$$

where $C := 2e^{||Q||} - 1$.

The necessity. We have that $L(X_n(1)) \Rightarrow L(X_0(1))$ and $\{\log (1 + ||X_n(1)||) : n \in \mathbb{N}\}$ is uniformly integrable. Using Lemma 3 we obtain that for every $r > 1$ and $k > 0$

(3) $\quad \{\log_r (1 + k \sup_{0 \le t \le 1} ||X_n(t)||) : n \in \mathbb{N}\}$ is uniformly integrable.

Further, $L(X_n(1)) \Rightarrow L(X_0(1))$ implies that $L(X_n) \Rightarrow L(X_0)$ in $D_E[0,t]$ for every $t>0$ (cf. Gikhman and Skorohod (1974), Theorem VI. 5.5) and Lemma 4 yields $L(Y_n(t)) \to L(Y_0(t))$ for every fixed $t \in [0,\infty)$ as $n \to \infty$. Moreover, $L(Y_0(t)) \Rightarrow L(Y_0(\infty))$ as $t \to \infty$. Therefore, to prove that $L(Y_n(\infty)) \to L(Y_0(\infty))$, as $n \to \infty$, it is enough to show that for each $\varepsilon > 0$

(4) $\quad \lim_{s\to\infty} \limsup_{n\to\infty} P\{||Y_n(\infty) - Y_n(s)|| > \varepsilon\} = 0,$

cf. Billingsley (1968), Theorem 4.2.

Since the function $t \to \|e^{-tQ}\|$ is submiltiplicative which vanishes at $+\infty$ there are positive constants a and b such that $\|e^{-tQ}\| \le ae^{-bt}$ for every $t \ge 0$. Let $q = e^{-b} < p < 1$ and $a_k := (1-p)p^k$, $k = 0,1,2,\ldots$. For a given $\varepsilon > 0$ and $m \in \mathbb{N}$ such that $a^{-1}\varepsilon(1-p)p^{-m} > 1$ we obtain

$$P\{\|Y_n(\infty) - Y_n(m)\| > \varepsilon\} = P\{\|\sum_{j=m}^{\infty}[Y_n(j+1) - Y_n(j)]\| > \varepsilon\}$$

$$\le P\{\sum_{j=m}^{\infty} aq^j\|\xi_{nj}\| > \sum_{j=m}^{\infty}\varepsilon a_{j-m}\} \le \sum_{j=m}^{\infty} P\{\|Y_n(1)\| > a^{-1}\varepsilon q^{-j}a_{j-m}\}$$

$$\le \sum_{j=m}^{\infty} P\{C \sup_{0\le t\le 1}\|X_n(t)\| > (p/q)^j\} \le \sum_{j=m}^{\infty} P\{\log_{p/q}(1 + C \sup_{0\le t\le 1}\|X_n(t)\|) > j\}$$

$$\le E[\log_{p/q}(1 + C \sup_{0\le t\le 1}\|X_n(t)\|)\, 1_{\{\log_{p/q}(1 + C \sup_{0\le t\le 1}\|X_n(t)\|) > m\}}.$$

Hence and from (3) we get

$$\lim_{m\to\infty} \sup_n P\{\|Y_n(\infty) - Y_n(m)\| > \varepsilon\} = 0$$

which implies (4). The proof of the necessity is complete.

The sufficiency. Note that

(5) $$Y_n(\infty) = Y_n(t) + [Y_n(\infty) - Y_n(t)],$$

$Y_n(\infty) - Y_n(t)$ is independent of $Y_n(t)$ and $Y_n(\infty) - Y_n(t) \overset{d}{=} e^{-tQ}Y_n(\infty)$. Since by our assumption $L(Y_n(\infty)) \Rightarrow L(Y_0(\infty))$ we obtain that the following sets are conditionally compact

$$\{L(Y_n(\infty) - Y_n(t)): 0 \le t \le \infty, n \in \mathbb{N}\}$$

and

$$\{L(Y_n(t)): 0 \le t \le \infty, n \in \mathbb{N}\}.$$

Using (5) and the fact that the characteristic functionals of infinitely divisible distributions are not vanishing, we obtain for every sequence $\{t_n\} \subset [0,\infty]$ such that $t_n \to t_0$ as $n \to \infty$

$$(6) \qquad \begin{cases} L(Y_n(t_n)) \Rightarrow L(Y_0(t_0)) \text{ and} \\ L(Y_n(\infty) - Y_n(t_n)) \Rightarrow L(Y_0(\infty) - Y_0(t_0)) \end{cases}$$

Since the conditional compactness of distributions implies the boundedness in probability of the corresponding r.v.'s we can find $r_0 \ge 1$ such that

$$(7) \qquad \sup_{n \in \mathbb{N}} \sup_{0 \le t \le \infty} P\{||Y_n(\infty) - Y_n(t)|| > r\} < \frac{1}{2} \quad \text{for all } r > r_0.$$

By Proposition 2 we get for any $a \ge 0$ and $n \in \mathbb{N}$

$$P\{\sup_{a \le t \le \infty} ||Y_n(t) - Y_n(a)|| > 2r\} \le 2P\{||Y_n(\infty) - Y_n(a)|| > r\}.$$

Hence

$$2P\{||Y_n(\infty) - Y_n(k)|| > r\} \ge P\{\sup_{k<m<\infty} ||Y_n(m) - Y_n(k)|| > 2r\}$$

$$= P\{\sup_{k<m} ||\sum_{j=k}^{m-1} [Y_n(j+1) - Y_n(j)]|| > 2r\} \ge$$

$$\ge P\{\sup_{k \le m} ||Y_n(m+1) - Y_n(m)|| > 4r\}$$

$$= 1 - \prod_{j=k}^{\infty} [1-P\{||Y_n(j+1) - Y_n(j)|| > 4r\}]$$

$$\ge 1 - \exp[-\sum_{j=k}^{\infty} P\{||Y_n(j+1) - Y_n(j)|| > 4r\}].$$

Therefore

$$\sum_{j=k}^{\infty} P\{||e^{-jQ}Y_n(1)|| > 4r\} \le -\log (1 - d_k),$$

where $d_k = d_k(r) := 2 \sup_{n\in\mathbb{N}} P\{||Y_n(\infty) - Y_n(k)|| > r\}$ and $r > r_0$.

Note now that $||e^Q|| > 1$. Indeed, this follows from the inequality $1 \le ||e^Q||^m\, ||e^{-mQ}||$ and the assumption $\lim_{m\to\infty} ||e^{-mQ}|| = 0$.

Thus

$$\sum_{j=k}^{\infty} P\{\log^+||Y_n(1)|| > \log 4r + j \log ||e^Q||\}$$

$$= \sum_{j=k}^{\infty} P\{||Y_n(1)|| > 4r\, ||e^Q||^j\} \le \sum_{j=k}^{\infty} P\{||e^{-jQ}Y_n(1)|| > 4r\} \le -\log (1-d_k).$$

By (6) for any fixed $r > 0$ $\lim_{k\to\infty} d_k = 0$. Using Lemma 1 we get that $\{\log (1 + ||Y_n(1)||): n \in \mathbb{N}\}$ is uniformly integrable. Since for $0 \le t \le 1$

$$Y_n(1) - Y_n(t) = \int_{(t,1]} e^{-sQ} dX_n(s) \overset{d}{=} e^{-tQ}Y_n(1-t) \tag{8}$$

and (6) we obtain by Lemma 3 that

$$\{\log (1 + \sup_{0\le t\le 1} ||Y_n(t)||): n \in \mathbb{N}\} \text{ is uniformly integrable.} \tag{9}$$

We have

$$e^{-Q}X_n(1) = \int_{(0,1]} e^{rQ} dY_n(r) = e^Q Y_n(1) - \int_0^1 Qe^{rQ}Y_n(r)dr. \tag{10}$$

Therefore $||X_n(1)|| \le C_2 \sup_{0\le t\le 1} ||Y_n(t)||$, where $C_2 := e^{||Q||}(2e^{||Q||}-1)$ and (9) implies that $\{\log (1 + ||X_n(1)||): n\in\mathbb{N}\}$ is uniformly integrable.

To complete the proof of the sufficiency it is enough to show that $L(X_n(1)) \Rightarrow L(X_0(1))$. In view of (10) and Lemma 4 it is sufficient to prove that $L(Y_n) \Rightarrow L(Y_0)$ in $D_E[0,1]$. Since Y_n, $n \in \mathbb{N}$, have independent increments and for $0 \le s < t \le 1$

$$Y_n(t) - Y_n(s) = \int_{(s,t]} e^{-rQ} dX_n(r) \overset{d}{=} e^{-sQ} Y_n(t - s),$$

(6) gives the weak convergence of all finite dimensional distributions of stochastic processes $\{Y_n(t): t \in [0,1]\}_{n\in\mathbb{N}}$. Moreover for every $\varepsilon > 0$

$$\lim_{h\to 0} \limsup_{n\to\infty} \sup_{|t-s|<h} P\{\|Y_n(t) - Y_n(s)\| > \varepsilon\}$$

$$\le \lim_{h\to 0} \limsup_{n\to\infty} \sup_{u<h} P\{\|Y_n(u)\| > C_3^{-1} \varepsilon\} = 0,$$

where $C_3 := \max_{0\le t\le 1} \|e^{-tQ}\|$. By Gikhman and Skorohod (1974) Theorem VI.5.5. $L(Y_n) \Rightarrow L(Y)$ in $D_E[0,1]$, which completes the proof of Theorem 1.

4. Proof of Theorem 2.

We begin this section with some auxilliary lemmas needed in the proof of Theorem 2.

Let Ψ be the class of all continuous non-decreasing functions $\psi:[0,\infty)\to[0,\infty)$ with $\psi(0) > 0$, $\lim_{t\to\infty} \frac{\psi(t)}{t} = \infty$ and such that $\psi(2t) \leq a\psi(t)$ for all $t \geq 0$ and some $a = a(\psi)$.

The following is a stronger version of the well-known criterium of the uniform integrability, cf. Meyer (1966), T22. We assume additionally that the constructed function ψ satisfies the so-called Δ_2-condition.

Lemma 5. *Let* $\{f_\alpha: \alpha \in I\}$ *be a family of measurable functions defined on a measurable space* $(S,\mathcal{S})$ *and let* $\{\nu_\alpha:\alpha \in I\}$ *be a family of measures on* $\mathcal{S}$ *such that* $\sup_{\alpha\in I} \nu_\alpha(S) < \infty$. *If*

(i) $$\lim_{t\to\infty} \sup_{\alpha\in I} \int_{\{|f_\alpha|>t\}} |f_\alpha| d\nu_\alpha = 0$$

then there exists a function $\psi \in \Psi$ *such that*

(ii) $$\sup_{\alpha\in I} \int_S (\psi \circ |f_\alpha|) d\nu_\alpha < \infty.$$

Conversely, if (ii) *is satisfied for some non-decreasing continuous function* $\psi: [0,\infty) \to [0,\infty)$ *with* $\lim_{t\to\infty} \frac{\psi(t)}{t} = \infty$, *then* (i) *holds.*

Proof. Following the proof of T22 in Meyer (1966) (with some obvious modifications) we show that (i) holds if and only if there exists a non-decreasing continuous function $\psi: [0,\infty) \to [0,\infty)$ with $\lim_{t\to\infty} \frac{\psi(t)}{t} = \infty$ such that $\sup_{\alpha\in I} \int_S (\psi\circ|f_\alpha|)d\nu_\alpha < \infty$. Put $\psi_1(t) = \psi(t) + 1$. Then $\sup_{\alpha\in I} \int_S (\psi_1\circ|f_\alpha|)d\nu_\alpha < \infty$.

To complete the proof it is enough to show that there exists $\psi_2 \in \Psi$ such that $\psi_2 \le \psi_1$.

Choose $c > 2$ such that $c \ge \sup_{0\le t\le 1} (\psi_1(t)/\psi_1(t/2))$ and define

$$\psi_2(t) := \psi_1(t) \quad \text{for } t \in [0,1]$$

and, by induction, put

$$\psi_2(t) := \min \{\psi_1(t),\ c\psi_2(t/2)\}$$

for $t \in [2^{n-1}, 2^n]$, $n = 1,2,\ldots$. In other words

$$\psi_2(t) = \min \{\psi_1(t),\ c\psi_1(t/2),\ldots,\ c^n\psi_1(t/2^n)\}$$

for $t \in [2^{n-1},2^n]$, $n = 1,2,\ldots$.

Clearly, ψ_2 is continuous, non-decreasing, $\psi_2(0) > 0$ and $\psi_2(2t) \le c\psi_2(t)$. It remains to show that $\lim_{t\to\infty} \psi_2(t)/t = \infty$. To this end observe that for every $n \ge 1$ there exists $0 \le j_n \le n$ such that $\psi_2(2^n) = c^{n-j_n}\psi_1(2^{j_n})$. Thus for $t \in [2^n,2^{n+1}]$ we have

$$\frac{\psi_2(t)}{t} \ge \frac{\psi_2(2^n)}{2^{n+1}} = \frac{1}{2}\left(\frac{c}{2}\right)^{n-j_n}\left(\frac{\psi_1(2^{j_n})}{2^{j_n}}\right) \to \infty$$

as $t \to \infty$, which completes the proof.

Lemma 6. *If $\phi \in \Phi$ and $\psi \in \Psi$, then $\psi \circ \phi \in \Phi$.*

Proof. Since $(\psi \circ \phi)(x + y) = \psi(\phi(x + y)) \le \psi(c\phi(x)\phi(y))$ it is enough to prove the submultiplicity of ψ with a constant. The condition $\psi(2t) \le a\psi(t)$ for every $t \ge 0$ yields $\psi(st) \le as^q\psi(t)$, where $q := \log_2 a$. Hence $\psi(1) = \psi(s^{-1}s) \le as^{-q}\psi(s)$ and consequently $a\psi(s)/\psi(1) \ge s^q$ for every $s \ge 0$. Finally we get $\psi(st) \le as^q\psi(t) \le (a^2/\psi(1))\psi(s)\psi(t)$, which completes the proof.

It is well-known that any infinitely divisible distribution with Lévy measure concentrated on some ball has all exponential moments finite. The following is a generalization of this result to sequences of infinitely divisible distributions.

Lemma 7. *Let a sequence* $\mu_n = [a_n, R_n, M_n]$, $n \in \mathbb{N}$, *be conditionally compact and for some* $r > 0$ $M_n(\{||x|| > r\}) = 0$ *for all* $n \in \mathbb{N}$. *Then for every* $\lambda > 0$

$$\sup_n \int_E \exp(\lambda ||x||)\mu_n(dx) < \infty.$$

Proof. Let $\gamma_n := [0, R_n, 0]$ and $\nu_n := [0,0,M_n]$. Then $\mu_n = \delta(a_n)*\gamma_n*\nu_n$ and the sequences $\{\delta(a_n): n \in \mathbb{N}\}$, $\{\gamma_n: n \in \mathbb{N}\}$ and $\{\nu_n: n \in \mathbb{N}\}$ are conditionally compact; cf. Araujo and Giné (1980), Theorem 1.4.9 and Corollary 3.4.6. Clearly $\sup_n ||a_n|| < \infty$. Also $\sup_n \int_E \exp(\varepsilon ||x||^2)\gamma_n(dx) < \infty$ for some $\varepsilon > 0$ (cf. e.g. Chevet (1983), Theorem 1(1)). Hence for every $\lambda > 0$ $\sup_n \int_E \exp(\lambda ||x||)\gamma_n(dx) < \infty$.

It remains to show that for every $\lambda > 0$

$$\sup_n \int_E \exp(\lambda ||x||)\nu_n(dx) < \infty. \tag{11}$$

Let us fix $\lambda > 0$ and for each ν_n construct an infinitesimal triangular array of random variables uniformly bounded by 2r which row sums weakly converge to ν_n; cf. the proof of Corollary 3.3 in de Acosta (1982). Now from the sequence of triangular arrays choose another one, say $\{Z_{nj}: 1 \le j \le k_n, n \ge 1\}$, such that

$$E \exp(\lambda ||S_n||) - \int_E \exp(\lambda ||x||)\nu_n(dx) \to 0$$

as $n \to \infty$ and $d(L(S_n), \nu_n) < \frac{1}{n}$, where $S_n = \sum_{j=1}^{k_n} Z_{nj}$ and d denotes the Prokhorov metric. Therefore $\{L(S_n)\ n \in \mathbb{N}\}$ is conditionally compact and Theorem 2.1 in de Acosta and Giné (1979) yields

$$\sup_n E \exp(\lambda||S_n||) < \infty.$$

This implies (11) and completes the proof.

Proof of Theorem 2. Let $M_n^1(B) := M(B\cap\{||x|| \leq 1\})$ and $M_n^2(B) := M(B\cap\{||x|| > 1\})$ for $B \in \mathcal{B}(E\setminus\{0\})$. Define also $\mu_n^1 := [a_n, R_n, M_n^1]$ and $\mu_n^2 := [0,0,M_n^2]$, $n \in \mathbb{N}$. Of course $\mu_n = \mu_n^1 * \mu_n^2$ and both $\{\mu_n^1\}$ and $\{\mu_n^2\}$ are conditionally compact; cf. Araujo and Giné (1980), Theorem 1.4.9 and Corollary 3.4.6. Further, note that for every function $\phi \in \Phi$ there are positive constants α and β such that

(12) $$\phi(x) \leq \alpha e^{\beta||x||}, \quad \text{for all } x \in E.$$

The sufficiency. Assume that

$$\lim_{t\to\infty} \sup_{n\in\mathbb{N}} \int_{\{\phi(x)>t\}} \phi(t)\mu_n(dx) = 0.$$

By Lemma 5 there exists $\psi \in \Psi$ such that $\sup_n \int_E(\psi \circ \phi)(x)\mu_n(dx) < \infty$. By Lemma 6 $\chi := \psi \circ \phi \in \Phi$ and by (12) $\chi(x) \leq \alpha e^{\beta||x||}$ for all $x \in E$ and some $\alpha,\beta > 0$. Since $\chi \in \Phi$ we have $\chi(y) = \chi(x + y - x) \leq c\chi(x + y)\chi(-x)$ for every $x,y \in E$ and some $c > 0$. Thus

$$e^{-\beta||x||}\chi(y) \leq \alpha c\chi(x + y) \quad \text{for all } x,y \in E.$$

Therefore,

(13) $$\int_E e^{-\beta||x||}\mu_n^1(dx) \int_E \chi(y)\mu_n^2(dy) \leq \alpha c \int_E \chi(z)\mu_n(dz)$$

and the quantity on the right-hand side is uniformly bounded for all $n \in \mathbb{N}$. By Lemma 7 $\sup_n \int_E e^{\beta||x||}\mu_n^1(dx) := K < \infty$. This together with the inequality

$$1 = \int_E \exp\left(\frac{\beta}{2}||x|| - \frac{\beta}{2}||x||\right)\mu_n^1(dx) \le \left(\int_E e^{\beta||x||}\mu_n^1(dx)\right)^{1/2}\left(\int_E e^{-\beta||x||}\mu_n^1(dx)\right)^{1/2}$$

implies that $\inf_{n\in\mathbb{N}} \int_E e^{-\beta||x||}\mu_n^1(dx) \ge K^{-1} > 0$. By (13) we get $\sup_{n\in\mathbb{N}} \int_E \chi(x)\mu_n^2(dx) < \infty$.

So, finally we obtain

$$\infty > \sup_{n\in\mathbb{N}} \int_E \chi(x)\mu_n^2(dx) = \sup_{n\in\mathbb{N}} \left[e^{-M_n^2(E)} \sum_{k=0}^{\infty} (k!)^{-1} \int_E \chi(x)(M_n^2)^{*k}(dx)\right]$$

$$\ge \sup_{n\in\mathbb{N}} \left[e^{-M_n^2(E)}\left(\chi(0) + \int_E \chi(x)M_n^2(dx)\right)\right].$$

Since $\{M_n^2\}$ is conditionally compact, cf. Araujo and Giné (1980), Theorem 3.4.5, $\sup_{n\in\mathbb{N}} M_n^2(E) < \infty$. Therefore $\sup_n \int_E \chi(x)M_n^2(dx) < \infty$. By Lemma 5 we get $\lim_{t\to\infty} \sup_{n\in\mathbb{N}} \int_{\{\phi(x)>t\}} \phi(x)M_n^2(dx) = 0$. Since $\{\phi(x) > t\} \subset \{||x|| > 1\}$ for sufficiently large $t > 0$ (cf. (12)) we obtain $\lim_{t\to\infty} \sup_{n\in\mathbb{N}} \int_{\{\phi(x)>t\}} \phi(x)M_n(dx) = 0$, which ends the proof of the sufficiency.

The necessity. Since we have that $\lim_{t\to\infty} \sup_{n\in\mathbb{N}} \int_{\{\phi(x)>t\}} \phi(x)M_n^2(dx) = 0$ and $\sup_{n\in\mathbb{N}} M_n^2(E) < \infty$ Lemma 5 shows that for some function $\psi \in \Psi$ $\sup_{n\in\mathbb{N}} \int_E \chi(x)M_n^2(dx) < \infty$, where $\chi := \psi \circ \phi$. By Lemma 6 $\chi(x + y) \le c\chi(x)\chi(y)$ for all $x, y \in E$ and some $c > 0$. Similarly as in de Acosta (1980), Corollary 3.4 we obtain

$$e^{M_n^2(E)}\int_E \chi(x)\mu_n^2(dx) = \sum_{k=0}^{\infty}(k!)^{-1}\int_E \chi(x)(M_n^2)^{*k}(dx)$$

$$= \chi(0) + \sum_{k=1}^{\infty}(k!)^{-1}\int_E\cdots\int_E \chi(x_1+\ldots+x_k)M_n^2(dx_1)\ldots M_n^2(dx_k)$$

$$\leq \chi(0) + \sum_{k=1}^{\infty}(k!)^{-1}\, c^{k-1}\left(\int_E \chi(x)M_n^2(dx)\right)^k$$

$$= \chi(0) + c^{-1}[\exp(c\int_E \chi(x)M_n^2(dx)) - 1].$$

Hence $\sup_n \int_E \chi(x)\mu_n^2(dx) < \infty$. Using (12) and Lemma 7 we obtain

$$\sup_{n\in\mathbb{N}}\int_E \chi(x)\mu_n(dx) \leq c\,\alpha \sup_{n\in\mathbb{N}}\int_E e^{\beta||x||}\mu_n^1(dx)\ \sup_{n\in\mathbb{N}}\int_E \chi(x)\mu_n^2(dx) < \infty,$$

which by Lemma 5 gives that $\lim_{t\to\infty}\sup_{n\in\mathbb{N}}\int_{\{\phi(x)>t\}}\phi(t)\mu_n(dx)=0$ and completes the proof of Theorem 2.

5. Proof of Theorem 3.

In the proof we shall use the following lemma:

Lemma 8. *For every* $[0,0,M] \in ID_{\log}$ *there exists a sequence* $M_n := \sum_{j=1}^{k_n} a_{nj}\delta(x_{nj})$, $n \in \mathbb{N}$, *where* $a_{nj} > 0$ *and* $x_{nj} \in E$, *such that* $[0,0,M_n] \Rightarrow_{\log} [0,0,M]$.

Proof. The proof of Lemma 8 is divided in three steps.

Step 1. Let $[0,0,M] \in ID_{\log}$ and $M_n(B) := M(B \cap \{||x|| > \frac{1}{n}\})$, $B \in \mathcal{B}(E)$, $n \in \mathbb{N}$. Then M_n are finite measures such that $[0,0,M_n] \Rightarrow_{\log} [0,0,M]$.

Proof. Theorem 3.4.7 in Araujo and Giné (1980) gives that $[0,0,M_n] \Rightarrow [0,0,M]$. Since $\sup_n \int_{||x||>t} \log(1 + ||x||)dM_n(x) \leq \int_{||x||>t} \log(1 + ||x||)dM(x) \to 0$ as $t \to \infty$, Theorem 2 completes the proof.

Step 2. Let M be a finite measure on E and $[0,0,M] \in ID_{\log}$. Then there are $M_n := \sum_{j=1}^{\infty} \alpha_{nj}\delta(x_{nj})$, where $a_{nj} > 0$, $\sum_{j=1}^{\infty} a_{nj} < \infty$ and $x_{nj} \in E$, such that $[0,0,M_n] \Rightarrow_{\log} [0,0,M]$.

Proof. Let for every n $\{A_{nj}: j \in \mathbb{N}\}$ be a partition of E onto non-empty Borel sets with $\operatorname{diam}(A_{nj}) \leq \frac{1}{n}$. Choose $x_{nj} \in A_{nj}$ and put $a_{nj} = M(A_{nj})$. Then $M_n \Rightarrow M$ and since $M(E) < \infty$, $[0,0,M_n] \Rightarrow [0,0,M]$. Furthermore for every $s>1$ we have

$$\int_{\{||x||>s\}} \log(1 + ||x||)M_n(dx) = \sum_{\{j:||x_{nj}||>s\}} \int_{A_{nj}} \log(1 + ||x_{nj}||)M(dx)$$

$$\leq \sum_{\{j:||x_{nj}||>s\}} \int_{A_{nj}} [\log(1 + ||x-x_{nj}||) + \log(1 + ||x||)]M(dx)$$

$$\leq (\log 2)M(\{||x||>s-1\}) + \int_{\{||x||>s-1\}} \log(1 + ||x||)M(dx).$$

Therefore Theorem 2 completes the proof of Step 2. Combining Steps 1 and 2 and the following obvious fact given below as Step 3 the proof of Lemma 8 follows.

Step 3. Suppose $M = \sum_{j=1}^{\infty} a_j\delta(x_j)$, with $x_j \in E$, $a_j > 0$ and $\sum_j a_j < \infty$. If $[0,0,M] \in ID_{log}$ then $[0,0,\sum_{j=1}^{n} a_j\delta(x_j)] \Rightarrow_{log} [0,0,M]$ as $n \to \infty$.

Proof of Theorem 3. Let $\mu = [a,R,M] \in ID_{log}$. Then $J_Q(\mu) := [a_\infty,R_\infty,M_\infty]$, where

(14) $$a_\infty = Q^{-1}a + \int_0^\infty\int_{E\setminus\{0\}} e^{-tQ} x [1_B(e^{-tQ}x)-1_B(x)]M(dx)dt,$$

(15) $$R_\infty = \int_0^\infty e^{-tQ}Re^{-tQ^*}dt$$

(16) $$M_\infty(F) = \int_0^\infty M(e^{tQ}F)dt, \qquad F \in \mathcal{B}(E\setminus\{0\}),$$

and $B = \{||x|| \le 1\}$; cf. Jurek (1982). From (15) we get that $QR_\infty + R_\infty Q^* = R$. Hence the operator $QR_\infty + R_\infty Q^*$ is non-negative and symmetric, provided R_∞ is the covariance operator of a Gaussian measure from $L(Q)$. Conversely, if R' is a Gaussian covariance operator such that $QR' + R'Q^*$ is non-negative, then $R' = \int_0^\infty e^{-tQ}(QR' + R'Q^*)e^{-tQ^*}dt$ and hence $R' \ge e^{-sQ}R'e^{-sQ^*}$ for all $s \ge 0$. Thus R' is the covariance operator of a Gaussian measure from $L(Q)$.

To complete the proof of Theorem 3, in view of Lemma 8 and Theorem 1, it is enough to show that for every $a \in E\setminus\{0\}$

(17) $$(\delta(a))_\infty = M_{\alpha,z}$$

for some $\alpha > 0$ and $z \in S_Q$ (note that $(cM)_\infty = cM_\infty$). It is easy to see that the mapping $\rho: S_Q \times \mathbf{R}^+ \to E\setminus\{0\}$ given by $\rho(u,t) := t^Q u$ is a homeomorphism.

Hence for every $a \in E\setminus\{0\}$ there exist $\alpha \in R^+$ and $z \in S_Q$ such that $a=\alpha^Q z$. Let $F = \rho(A\times B)$, where $A\in\mathcal{B}(S_Q)$ and $B \in \mathcal{B}(\mathbb{R}^+)$. By (16) we obtain

$$(\delta(a))_\infty(F) = \int_0^\infty \delta(\alpha^Q z)\{s^Q u:\ u \in A,\ s \in e^t B\}dt$$

$$= 1_A(z)\int_{\{t:e^{-t}\alpha\in B\}} = \delta(z)(A) \int_0^\alpha 1_B(s)s^{-1}ds$$

$$= \int_{S_Q}\int_0^\alpha 1_A(x)1_B(s)s^{-1}ds\delta(z)(dx)$$

$$= \int_{S_Q}\int_0^\alpha 1_F(s^Q x)s^{-1}ds\delta(z)(dx)$$

$$= \int_0^\alpha 1_F(s^Q z)s^{-1}ds.$$

This proves $(\delta(a))_\infty(F)= M_{\alpha,z}(F)$ for $F = \rho(A \times B)$. Since ρ is a homeomorphism this equality extends to all Borel sets $F \subset E\setminus\{0\}$, which proves (17) and completes the proof.

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